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METELITSYN'S METHOD AND THEOREMS†

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Editorial note. This paper and the next paper by Yu. K. Zhbanov and V. F. Zhuravlev are concerned with the following theorems due to I. I. Metelitsyn (1952):

Theorem 1. If a conservative system is statically unstable, the system cannot be made stable by adding strictly non-conservative forces (without dissipative and gyroscopic forces).

Theorem 2. If a conservative system is statically stable, the addition of strictly non-conservative forces (without dissipative and gyroscopic forces) can make the system unstable.

Theorem 3. A strictly non-conservative system ($V \equiv 0$) can be made stable only if gyroscopic and dissipative forces are simultaneously added to the active forces.

Theorem 4. A statically unstable system can be made stable if dissipative, gyroscopic and strictly non-conservative forces are simultaneously added to the forces applied.

Theorem 5. If the condition of stability $TE^2 - \Gamma DE < D^2V$ is satisfied and gyroscopic forces predominate over the others, then the vibration frequencies of the system diverge, i.e. some of them become extremely small, while others become extremely large.

Theorem 6. If the stability condition is satisfied and gyroscopic forces predominate, then more intense damping compared with that of slow vibrations corresponds to vibrations with higher frequencies.

Theorem 7. If, among the roots of the characteristic equation, there are real roots, then stability of motion is only possible when the potential energy is a positive-definite quadratic form.

Here, T corresponds to the kinetic energy, D to dissipative forces, Γ to gyroscopic forces, V to potential forces and E to non-conservative positional forces.

Up to the 1920s, systems containing potential, dissipative and gyroscopic forces were considered in mechanics, although the existence of non-conservative positional forces had been known for a long time. In 1928, Nikolai appears to have been the first to establish that a following force acting on a bent elastic rod can be split into potential and non-conservative positional forces [1]. This caused great interest, and many papers appeared (their number increased sharply with the development of jet propulsion technology) in which systems acted upon by potential, dissipative and non-conservative positional forces were considered (see, for example, [2]).

In 1952, Metelitsyn [3] examined in general form a system containing all existing linear forces, namely: potential, dissipative, gyroscopic and non-conservative positional. A concise version of his paper is contained in [4]; it is included without change in [5]. The general formulation of the problem aroused considerable interest, which is largely to Metelitsyn's credit.

To investigate his system, Metelitsyn used a method that was known as far back as the second half of the nineteenth century [6]. This method had been used to investigate systems containing only potential and dissipative forces – Metelitsyn extended its range of application.

The editors of Metelitsyn's collected papers [5] remarked at the very start of the paper in question that Metelitsyn "by stability always means potential stability, and by a static unstable system he means a system for which the potential energy has a maximum at the equilibrium position" – these remarks will be taken into account below.

There are at present different opinions concerning Metelitsyn's work. His supporters continue to publish papers based on his work without scrutinizing the special features of his method. It is therefore useful to explain Metelitsyn's theorems and the method he used in detail.

Metelitsyn examines a system whose motion is described by linear homogeneous second-order differential equations with constant coefficients.

$$\sum_{s=1}^{n}(\alpha_{ks}\ddot{q}_s + \beta_{ks}\dot{q}_s + \gamma_{ks}\dot{q}_s + \delta_{ks}q_s + \varepsilon_{ks}q_s) = 0, \quad k = 1,\ldots,n \tag{1}$$

where $\alpha_{ks} = \alpha_{sk}$ are the coefficients of inertia that occur in the kinetic energy of the system, $\beta_{ks}\dot{q}_s$ are dissipative forces ($\beta_{ks} = \beta_{sk}$), $\gamma_{ks}\dot{q}_s$ are gyroscopic forces ($\gamma_{ks} = -\gamma_{sk}$), $\delta_{ks}q_s$ are potential forces ($\delta_{ks} = \delta_{sk}$) and $\varepsilon_{ks}q_s$ are non-conservative positional forces ($\varepsilon_{ks} = -\varepsilon_{sk}$). Note that the latter forces have different names (Metelitsyn calls them *strictly non-conservative forces*). Here, all the notation of [5] is retained.

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For greater clarity, using a linear orthogonal transformation we will reduce Eqs (1) to normal coordinates, where, when there are no forces apart from potential forces, these equations take the form $\ddot{q}_k + \delta_k q_k = 0$. As a result, instead of Eqs (1), we obtain the equivalent equations (without loss of generality, the former notation is retained)

$$\sum_{s=1}^{n}(\ddot{q}_k + \beta_{ks}\dot{q}_s + \gamma_{ks}\dot{q}_s + \delta_k q_k + \varepsilon_{ks}q_s) = 0, \qquad k = 1,\ldots,n \tag{2}$$

As usual, Metelitsyn seeks a solution in the form

$$q_s = A_s e^{\mu t} \tag{3}$$

After substitution, equations are obtained for determining the numbers A_s and μ

$$\sum_{s=1}^{n}[\mu^2 + (\beta_{ks} + \gamma_{ks})\mu + \delta_k + \varepsilon_{ks}]A_s = 0, \qquad k = 1,\ldots,n \tag{4}$$

$$\det \| \mu^2 + (\beta_{ks} + \gamma_{ks})\mu + \delta_k + \varepsilon_{ks} \| = 0 \tag{5}$$

Metelitsyn then assumes that the characteristic equation (5) has complex conjugate roots μ and μ', each of which has its own system of complex conjugate constants A_s and A_s'. Then, each equation of (4) is multiplied by A_k', and, after addition, the quadratic equation

$$T\mu^2 + (D + i\Gamma)\mu + V + iE = 0 \tag{6}$$

is obtained, where T, D, Γ, V and E are the sums and differences of quadratic forms in which, instead of variables, there are parameters which depend ultimately on the complex conjugate roots μ and μ'. The number T is always positive; for resistance forces with complete dissipation, $D > 0$. The quantities Γ and E can have any sign and are equal to zero. Note that Metelitsyn always refers to $T, \ldots, E$ as "quantities". In their construction, these quantities depend on the coefficients of Eq. (2), the numbers A_s and A_s' and the roots μ and μ'. The coordinates q and their time derivatives $\dot{q}$ do not occur in the quantities $T, \ldots, E$.

Assuming that the resistance forces have complete dissipation, Metelitsyn obtained the inequality

$$TE^2 - \Gamma DE < D^2 V \tag{7}$$

from the condition that the real part of the roots of Eq. (6) is negative. Metelitsyn calls inequality (7) "the condition of [asymptotic] stability of non-conservative systems".

Here, several general remarks must be made:

1. Taking into account that each pair of roots has its own system of quantities $T, \ldots, E$ (Metelitsyn himself mentions this [5, p. 40], it is necessary to require that inequality (7) be satisfied for all pairs of roots; this inequality is identical in form for all pairs of roots.

2. The quantity $V = \Sigma\delta_k A_k A_k'$ corresponds to the potential forces. If, for several but not all roots of Eq. (5), the values of V are, for example, positive, then this does not mean that the potential energy $\Pi = \frac{1}{2}\Sigma\delta_k q_k^2$ as a quadratic form of the coordinates will have a minimum at the equilibrium position.

3. Without solving Eq. (5), it is impossible to determine the numbers A_s and A_s' and, consequently, impossible to determine the quantities $T, \ldots, E$, and without these it is impossible to verify condition (7). In Theorems 5 and 6, Metelitsyn considers this condition to be satisfied. How this was checked is unknown.

4. Metelitsyn considered not only asymptotic stability. In Theorems 1 and 2 the stability of systems when there are no dissipative forces is spoken of. Under these conditions, only simple stability is possible.

We will now consider Metelitsyn's theorems.

Theorem 1. Metelitsyn's proof is as follows: "in fact, if $D \equiv 0$ and $\Gamma \equiv 0$, then the condition $TE^2 < 0$ cannot be satisfied".

In deriving inequality (7), Metelitsyn assumes that $D > 0$; here he assumes that $D = 0$.

We will now take into account the second remark of the editors. Suppose that, when a conservative system is unstable, the potential energy has a maximum. Then, in Eq. (5) and $\delta_k < 0$, and when $\beta_{ks} = 0$ and $\gamma_{ks} = 0$ the coefficient of μ^{2n-2} will be equal to $\Sigma\delta_k < 0$, which demonstrates the impossibility of the system being stabilized by any non-conservative positional forces. Thus, Theorem 1, with the reservation given by the editors, becomes correct, but its proof bears no relation to Metelitsyn's method and does not stem from it.

It is now necessary to examine this theorem from the point of view of stability theory, according to which, with time, Lagrangian motion is considered to be unstable if just one coordinate increases without limit. We will show, for example, that in this case Theorem 1 is incorrect. In fact, a system whose motion is described by the equations

$$\ddot{q}_1 + 6q_1 + \varepsilon q_2 = 0, \quad \ddot{q}_2 - q_2 - \varepsilon q_1 = 0$$

when $\varepsilon = 0$ is an unstable potential system. We will now assume that $\varepsilon = \sqrt{10}$. The characteristic equation will have pure imaginary roots: $\pm 2i$ and $\pm i$. This demonstrates that an unstable potential system can be stabilized by some non-conservative positional forces. The stability achieved, of course, is simple, since, without dissipative forces, asymptotic stability is unachievable – a fact that has long been known.

Theorem 2. Metelitsyn gives no proof, and no proof follows from inequality (7). This assumption does not look like a theorem, since the formulation enables the opposite assumption to be made, namely: *the addition of non-conservative positional forces is unable to disrupt the stability of a potential system*. The correctness of both assertions can easily be shown using examples. In any case, it follows from this theorem that Metelitsyn did not only consider asymptotic stability, since without dissipative forces only simple stability is possible.

Theorem 3. If, in this and in the following three theorems, Metelitsyn's assumption that inequality (7) is correct is accepted, then in Theorem 3 the condition *the determinant of strictly non-conservative forces must be non-zero*, does not suffice. This addition follows directly from Eq. (5), since under the conditions of the theorem the free term of this equation is equal to the determinant $\det\|\varepsilon_{ks}\|$. This determinant is skew-symmetric and consequently the case of old n is ruled out straight away; however, for even $n \geqslant 4$ this determinant may be equal to zero.

Theorem 4. It is assumed, as the editors of [5] write, that the potential energy has a maximum at the position of unstable equilibrium. Under these conditions, the theorem is correct only for even n, since for odd n the free term of the characteristic equation (5) will be negative. This addition is not present in Metelitsyn's case. The complete proof can be found in [7].

Theorem 5. The assertion that, when gyroscopic forces predominate, the vibration frequencies diverge is *correct* but the proof is incorrect. In fact, as follows from the proof of this theorem, by gyroscopic predominance Metelitsyn means the case when $\Gamma \gg \{T, D, V, E\}$, but even before this theorem (see [5, p. 4]) in a similar case he writes that this condition is necessary but insufficient, "since Γ can equal zero". Consequently, this condition is insufficient to prove Theorem 5. Furthermore, Metelitsyn took no account of the fact that the frequencies are determined from characteristic equation (5) and for the frequencies to diverge it is necessary for the determinant $G = \det\|\gamma_{ks}\|$ to be zero (see [7], where the necessary and sufficient conditions for the frequencies to diverge are stipulated and where there is a reference to an earlier publication).

Theorem 6. The remarks concerning Theorem 5 also apply to Theorem 6.

Theorem 7. Metelitsyn's proof: "in fact, in this case $\Gamma = 0$ and $E = 0$, and therefore condition (11) [inequality (7) here] reduces to the form $V > 0$". Note that Metelitsyn correctly shows ([5, p. 40]) that, for a real roof of Eq. (5), $\Gamma \equiv 0$ and $E \equiv 0$.

In this theorem Metelitsyn forgot that he always called $T, \ldots, E$ quantities which depend in a complex manner on the roots μ and μ', and he called V a quadratic form which depends on the coordinates. The consequences of this forgetfulness will be demonstrated by an example.

Suppose the equations of motion of the system have the form

$$\ddot{q}_1 + 3.03\dot{q}_1 - q_1 + 2\dot{q}_2 + \sqrt{5}q_2 = 0$$

$$-2\dot{q}_1 - \sqrt{5}q_1 + \ddot{q}_2 + 1.97\dot{q}_2 + q_2 = 0$$

All forces are acting on the system, and the characteristic equation has the roots (with an accuracy to 1%)

$$\mu_1 = -1, \qquad \mu_2 = -2, \qquad \mu_{3,4} = -1 \pm i$$

Two positive values of V correspond to the two real roots, but according to Remark 2 this is insufficient for the potential energy, as a function of the coordinates, to have a minimum at the equilibrium position. In fact, the system is asymptotically stable and has two positive values of V, and its potential energy, equal to $\Pi = (q_1^2 - q_2^2)/2$, contradicts Theorem 7.

We will return to inequality (7). Metelitsyn writes that this inequality "... expresses the condition of [asymptotic] stability of non-conservative systems".

All the quantities in inequality (7) can be determined only after the roots of the characteristic equation of the initial differential equations have been found, and here, for each root, its own inequality (7) should exist.

For simplicity we will take a system of two equations containing all forces,

$$\ddot{q}_1 + \beta_1\dot{q}_1 + \delta_1 q_1 + \gamma\dot{q}_2 + \varepsilon q_2 = 0 \tag{8}$$

$$-(\gamma\dot{q}_1 + \varepsilon q_1) + \ddot{q}_2 + \beta_2\dot{q}_2 + \delta_2 q_2 = 0$$

Assuming, as usual, that $q_s = A_s e^{\mu t}$, after substitution and cancellation of $e^{\mu t}$ we obtain

$$(\mu^2+\beta_1\mu+\delta_1)A_1+(\gamma\mu+\varepsilon)A_2=0$$
$$-(\gamma\mu+\varepsilon)A_1+(\mu^2+\beta_2\mu+\delta_2)A_2=0 \tag{9}$$

Following Metelitsyn, let us assume that

$$A_s=M_s+N_s i, \qquad s=1,2 \tag{10}$$

We substitute these equalities into system (9), and then multiply the first equation by $M_1-N_1 i$, and the second equation by $M_2-N_2 i$. Adding the expressions obtained, we obtain Eq. (6), in which

$$T=\sum(M_k^2+N_k^2), \quad D=\sum\beta_k(M_k^2+N_k^2), \quad V=\sum\delta_k(M_k^2+N_k^2)$$
$$\Gamma=2\gamma(M_1N_2-M_2N_1), \quad E=2\varepsilon(M_1N_2-M_2N_1) \tag{11}$$

Equalities (10) and (11) correspond to Eqs (6) and (8) in Metelitsyn's paper [5] when $n=2$.

If the characteristic equation is set up for system (8)

$$\Delta(\mu)=\begin{vmatrix}\mu^2+\beta_1\mu+\delta_1 & \gamma\mu+\varepsilon\\ -(\gamma\mu+\varepsilon) & \mu^2+\beta_2\mu+\delta_2\end{vmatrix}=0$$

then the constants A_s will be equal to the cofactors Δ_{1s} of the determinant $\Delta(\mu)$ (see [8]). In the present case

$$A_1=\mu^2+\beta_2\mu+\delta_2, \quad A_2=\gamma\mu+\varepsilon$$

Assuming that $\mu=\alpha+\omega i$ and taking equalities (10) into account, we obtain

$$M_1=\alpha^2-\omega^2+\beta_2\alpha+\delta_2, \quad N_1=\omega(2\alpha+\beta_2), \quad M_2=\gamma\alpha+\varepsilon, \quad N_2=\omega\gamma \tag{12}$$

Let us consider an example. When

$$\beta_1=6-\beta_2=5.8185891, \qquad \delta_1=\delta_2=-{}^1/_2, \qquad \gamma=11/3, \qquad \varepsilon={}^9/_4 \tag{13}$$

system (8) contains all forces, has complete dissipation and is asymptotically stable – the roots of the characteristic equation are as follows:

$$\mu_{1,2}=-1\pm0.5i, \qquad \mu_{3,4}=-2\pm0.5i$$

(All calculations were carried out to eight significant figures, with rounding up of the final result.)

Consider the first two roots $\mu_{1,2}$, for which

$$\alpha=-1, \qquad \omega=\pm0.5 \tag{14}$$

Using equalities (12) and (13), we find M_1, N_1, M_2 and N_2, and then, from formulae (11), we obtain

$$T=6.2000, \qquad D=5.8121, \qquad V=-3.1000, \qquad \Gamma=\mp8.5244, \qquad E=\mp5.2309$$

(the upper signs correspond to the upper sign of ω in (14), and the lower signs to the lower sign of ω).

Now, after elementary calculations, inequality (7) takes the form

$$169.6-259.2<-104.7$$

which for the roots $\mu_{1,2}$ contradicts Metelitsyn's assertion.

Repeating all the transformations for the roots $\mu_{3,4}$, we obtain the inequality

$$16237-48194<-115860$$

which again contradicts Metelitsyn's assertion.

This example (which, of course, is not the only one) shows that Metelitsyn was wrong – inequality (7) cannot be used as a criterion of asymptotic stability of the system of initial differential equations (1). This inequality serves as the condition for the real parts of the roots of quadratic equation (6) to be negative. The fact that the coefficients of this equation were obtained from the roots of Eq. (5) proves nothing, since the latter were assumed to be *only* complex-conjugate roots.

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